5.5 THE FAMILY OF QUADRATIC FUNCTIONS

In Chapter 2, we looked at the example of a baseball which is popped up by a batter. The height of the ball above the ground was modeled by the quadratic function

\[ y = f(t) = -16t^2 + 64t + 3, \]

where \( t \) is time in seconds after the ball leaves the bat, and \( y \) is in feet. The function is graphed in Figure 5.52.

![Figure 5.52](https://example.com/figure5.52.png)

**FIGURE 5.52** Height of baseball at time \( t \)

The point on the graph with the largest \( y \) value appears to be (2, 67). (We show this in Example 5 on page 219.) This means that the baseball reaches its maximum height of 67 feet 2 seconds after being hit. The maximum point (2, 67) is called the vertex.

The graph of a quadratic function is called a parabola; its maximum (or minimum, if the parabola opens upward) is the vertex.

**The Vertex of a Parabola**

The graph of the function \( y = x^2 \) is a parabola with vertex at the origin. All other functions in the quadratic family turn out to be transformations of this function. Let’s first graph a quadratic function of the form

\[ y = a(x-h)^2 + k \]

and locate its vertex.

**Example 1**

Let \( f(x) = x^2 \) and \( g(x) = -2(x+1)^2 + 3 \).

(a) Express the function \( g \) in terms of the function \( f \).

(b) Sketch a graph of \( f \). Transform the graph of \( f \) into the graph of \( g \).

(c) Multiply out and simplify the formula for \( g \).

(d) Explain how the formula for \( g \) can be used to obtain the vertex of the graph of \( g \).

**Solution**

(a) Since \( f(x+1) = (x+1)^2 \), we have

\[ g(x) = -2(x+1)^2 + 3 \]

(b) The graph of \( f(x) = x^2 \) is shown at the left in Figure 5.53. The graph of \( g \) is obtained from the graph of \( f \) in four steps, as shown in Figure 5.53.

(c) Multiplying out gives \( g(x) = -2x^2 - 4x - 2 + 3 = -2x^2 - 4x + 1 \)

(d) The vertex of the graph of \( f \) is (0, 0). In Step 1 the vertex shifts 1 unit to the left (because of the \((x+1)\) in the formula), and in Step 4 the vertex shifts 3 units up (because of the +3 in the formula). Thus, the vertex of the graph of \( g \) is at (-1, 3).

![Figure 5.53](https://example.com/figure5.53.png)

**FIGURE 5.53** The graph of \( f(x) = x^2 \), on the left, is transformed in four steps into the graph of \( g(x) = -2(x+1)^2 + 3 \), on the right

In general, the graph of \( g(x) = a(x-h)^2 + k \) is obtained from the graph of \( f(x) = x^2 \) by shifting horizontally \( |h| \) units, stretching vertically by a factor of \( a \) (and reflecting about the \( x \)-axis if \( a < 0 \)), and shifting vertically \( |k| \) units. In the process, the vertex is shifted from (0, 0) to the point \((h, k)\). The graph of the function is symmetrical about a vertical
Formulas for Quadratic Functions

The function \( g \) in Example 1 can be written in two ways:

\[
g(x) = -2(x + 1)^2 + 3
\]

and

\[
g(x) = -2x^2 - 4x + 1
\]

The first version is helpful for understanding the graph of the quadratic function and finding its vertex. In general, we have the following:

The **standard form** for a quadratic function is

\[
y = ax^2 + bx + c, \text{ where } a, b, c \text{ are constants, } a \neq 0
\]

The **vertex form** is

\[
y = a(x - h)^2 + k, \text{ where } a, h, k \text{ are constants, } a \neq 0
\]

The graph of a quadratic function is called a **parabola**. The parabola

- Has vertex \((h, k)\)
- Has axis of symmetry \(x = h\)
- Opens upward if \(a > 0\) or downward if \(a < 0\)

Thus, any quadratic function can be expressed in both standard form and vertex form. To convert from vertex form to standard form, we multiply out the squared term. To convert from standard form to vertex form, we **complete the square**.

**Example 2**

Put these quadratic functions into vertex form by completing the square and then graph them.

(a) \( s(x) = x^2 - 6x + 8 \)

(b) \( t(x) = -4x^2 - 12x - 8 \)

**Solution**

(a) To complete the square, \(^2\) find the square of half of the coefficient of the \(x\)-term, \((-6/2)^2 = 9\). Add and subtract this number after the \(x\)-term:

\[
s(x) = x^2 - 6x + 9 - 9 + 8
\]

so

\[
s(x) = (x - 3)^2 - 1
\]

The vertex of \(s\) is \((3, -1)\) and the axis of symmetry is the vertical line \(x = 3\). There is no vertical stretch since \(a = 1\), and the parabola opens upward. See Figure 5.54.

(b) To complete the square, first factor out \(-4\), the coefficient of \(x^2\), giving

\[
t(x) = -4(x^2 + 3x) + 2
\]

Now add and subtract the square of half the coefficient of the \(x\)-term, \((3/2)^2 = 9/4\), inside the parentheses. This gives

\[
t(x) = -4\left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + 2
\]

\[
t(x) = -4\left(x + \frac{3}{2}\right)^2 + 1
\]

The vertex of \(t\) is \((-3/2, 1)\), the axis of symmetry is \(x = -3/2\), the vertical stretch factor is 4, and the parabola opens downward. See Figure 5.55.
Finding a Formula From a Graph

If we know the vertex of a quadratic function and one other point, we can use the vertex form to find its formula.

**Example 3**

Find the formula for the quadratic function graphed in Figure 5.56.

**Solution**

Since the vertex is given, we use the form \( m(x) = a(x - h)^2 + k \) and find \( a \), \( h \), and \( k \). The vertex is \((-3, 2)\), so \( h = -3 \) and \( k = 2 \). Thus, \( m(x) = a(x - (-3))^2 + 2 \).

To find \( a \), use the \( y \)-intercept \((0, 5)\). Substitute \( x = 0 \) and \( y = m(0) = 5 \) into the formula for \( m(x) \) and solve for \( a \):

\[
\begin{align*}
5 &= a(0 + 3)^2 + 2 \\
3 &= 9a \\
a &= \frac{3}{9} = \frac{1}{3}.
\end{align*}
\]

Thus, the formula is

\[ m(x) = \frac{1}{3}(x + 3)^2 + 2. \]

If we want the formula in standard form, we multiply out:

\[ m(x) = \frac{1}{3}x^2 + 2x + 5. \]

**Example 4**

Find the equation of the parabola in Figure 5.57 using the factored form.
Solution

Since the parabola has x-intercepts at $x = 1$ and $x = 3$, its formula can be written as

$$y = a(x-1)(x-3)$$

Substituting $x = 0$, $y = 6$ gives

$$6 = a(3)$$
$$a = 2$$

Thus, the equation is

$$y = 2(x-1)(x-3)$$

Applications of Quadratic Functions

In applications, it is often useful to find the maximum or minimum value of a quadratic function. First, we return to the baseball example which started this section.

Example 5

For $t$ in seconds, the height of a baseball in feet is given by the formula

$$y = f(t) = -16t^2 + 64t + 3$$

Using algebra, find the maximum height reached by the baseball and the time at which the ball reaches the ground.

Solution

To find the maximum height, complete the square to find the vertex:

$$y = f(t) = -16(t^2 - 4t) + 3$$
$$= -16(t^2 - 4t + 4 - 4) + 3$$
$$= -16(t^2 - 4t + 4) - 16(-4) + 3$$
$$= -16(t - 2)^2 + 64 + 3$$
$$= -16(t - 2)^2 + 67$$

Thus, the vertex is at the point $(2, 67)$. This means that the ball reaches its maximum height of 67 feet at $t = 2$ seconds.

The time at which the ball hits the ground is found by solving $f(t) = 0$. We have

$$-16(t^2 - 4t + 4) = 0$$
$$t - 2 = \pm \frac{\sqrt{166}}{16}$$
$$t = 2 \pm 2.046$$

The solutions are $t \approx -0.046$ and $t \approx 4.046$. Since the ball was thrown at $t = 0$, we want $t \geq 0$. Thus, the ball hits the ground approximately 4.046 seconds after being hit.

Example 6

A city decides to make a park by fencing off a section of riverfront property. Funds are allotted to provide 80 meters of fence. The area enclosed will be a rectangle, but only three sides will be enclosed by fence—the other side will be bound by the river. What is the maximum area that can be enclosed in this way?

Solution

Two sides are perpendicular to the bank of the river and have equal length, which we call $h$. The other side is parallel to the bank of the river. Call its length $b$. See Figure 5.58. Since the fence is 80 meters long,

$$2h + b = 80$$
$$b = 80 - 2h$$
The area of the park, \( A \), is the product of the lengths of two adjacent sides, so

\[
A = bh = (50 - 2h) h = -2h^2 + 50h.
\]

The function \( A = -2h^2 + 80h \) is quadratic. Since the coefficient of \( h^2 \) is negative, the parabola opens downward and we have a maximum at the vertex. The zeros of this quadratic function are \( h = 0 \) and \( h = 40 \), so the axis of symmetry, which is midway between the zeros, is \( h = 20 \). The vertex of a parabola occurs on its axis of symmetry. Thus, substituting \( h = 20 \) gives the maximum area:

\[
A = (80 - 2 \cdot 20) \cdot 20 = 80 \cdot 20 = 1600 \text{ square meters}.
\]

**Exercises and Problems for Section 5.5**

**Exercises**

For the quadratic functions in Exercises 1 and 2, state the coordinates of the vertex, the axis of symmetry, and whether the parabola opens upwards or downwards.

1. \( f(x) = 3(x - 1)^2 + 2 \)
2. \( g(x) = -(x + 3)^2 - 4 \)
3. Sketch the quadratic functions given in standard form. Identify the values of the parameters \( a, b, \) and \( c \). Label the zeros, axis of symmetry, vertex, and \( y \)-intercept.
   - (a) \( g(x) = x^2 + 3 \)
   - (b) \( f(x) = -2x^2 + 4x + 16 \)
4. Find the vertex and axis of symmetry of the graph of \( v(t) = t^2 + 11t - 4 \).
5. Find the vertex and axis of symmetry of the graph of \( w(x) = -3x^2 - 30x + 11 \).
6. Show that the function \( y = -x^2 + 7x - 13 \) has no real zeros.
7. Find the value of \( k \) so that the graph of \( y = (x - 3)^2 + k \) passes through the point \( (6, 13) \).
8. The parabola \( y = ax^2 + k \) has vertex \((0, -2)\) and passes through the point \((3, 4)\). Find its equation.

In Exercises 9, 10, 11, 12, 13, and 14, find a formula for the parabola.
12. For Exercises 15, 16, 17 and 18, convert the quadratic functions to vertex form by completing the square. Identify the vertex and the axis of symmetry.

15. \( f(x) = x^2 + 8x + 3 \)

16. \( g(x) = -2x^2 + 12x + 4 \)

17. Using the vertex form, find a formula for the parabola with vertex (2, 5) which passes through the point (1, 2).

18. Using the factored form, find the formula for the parabola whose zeros are \( x = -1 \) and \( x = 5 \), and which passes through the point (2, 6).

Problems

In Problems 19, 20, 21, 22, 23 and 24, find a formula for the quadratic function whose graph has the given properties.

19. A vertex at (4, 2) and a \( y \)-intercept of \( y = 6 \).

20. A vertex at (4, 2) and a \( y \)-intercept of \( y = -4 \).

21. A vertex at (4, 2) and zeros at \( x = -3, 11 \).

22. A \( y \)-intercept of \( y = 7 \) and \( x \)-intercepts at \( x = 1, 4 \).

23. A \( y \)-intercept of \( y = 7 \) and one zero at \( x = -2 \).

24. A vertex at (−3, 7) and contains the point (−3, −7).

25. Graph \( y = x^2 - 10x + 25 \) and \( y = x^2 \). Use a shift transformation to explain the relationship between the two graphs.

26. (a) Graph \( h(x) = -2x^2 - 8x - 8 \).
   
   (b) Compare the graphs of \( h(x) \) and \( f(x) = x^2 \). How are these two graphs related? Be specific.

27. Let \( f \) be a quadratic function whose graph is a concave up parabola with a vertex at (1, −1), and a zero at the origin.

   (a) Graph \( y = f(x) \).
   
   (b) Determine a formula for \( f(x) \).
   
   (c) Determine the range of \( f \).
   
   (d) Find any other zeros.

28. Let \( f(x) = x^2 \) and let \( g(x) = (x - 3)^2 + 2 \).

   (a) Give the formula for \( g \) in terms of \( f \), and describe the relationship between \( f \) and \( g \) in words.
   
   (b) Is \( g \) a quadratic function? If so, find its standard form and the parameters \( a, b, \) and \( c \).
   
   (c) Graph \( g \), labeling all important features.

29. If we know a quadratic function \( f \) has a zero at \( x = -1 \) and vertex at (1, 4), do we have enough information to find a formula for this function? If your answer is yes, find it; if not, give your reasons.

30. Gwendolyn, a pleasant parabola, was taking a peaceful nap when her dream turned into a nightmare: she dreamt that a low-flying pterodactyl was swooping toward her. Startled, she flipped over the horizontal axis, darted up (vertically) by three units, and to the left (horizontally) by two units. Finally she woke up and realized that her equation was \( y = -(x - 1)^2 + 3 \). What was her equation before she had the bad dream?

31. A tomato is thrown vertically into the air at time \( t = 0 \). Its height, \( d(t) \) (in feet), above the ground at time \( t \) (in seconds) is given by

   \[ d(t) = -16t^2 + 4t + 10 \]

   (a) Graph \( d(t) \).
   
   (b) Find \( t \) when \( d(t) = 0 \). What is happening to the tomato the first time \( d(t) = 0 ? \) The second time?
   
   (c) When does the tomato reach its maximum height?
   
   (d) What is the maximum height that the tomato reaches?

32. An espresso stand finds that its weekly profit is a function of the price, \( x \), it charges per cup. If \( x \) is in dollars, the weekly profit is

   \[ P(x) = -2900x^2 + 7250x - 2900 \text{ dollars} \]

   (a) Approximate the maximum profit and the price per cup that produces that profit.
(b) Which function, \( P(x - 2) \) or \( P(x) - 2 \), gives a function that has the same maximum profit? What price per cup produces that maximum profit?

(c) Which function, \( P(x + 50) \) or \( P(x) + 50 \), gives a function where the price per cup that produces the maximum profit remains unchanged? What is the maximum profit?

33. If you have a string of length 50 cm, what are the dimensions of the rectangle of maximum area that you can enclose with your string? Explain your reasoning. What about a string of length \( k \) cm?

34. A football player kicks a ball at an angle of 37° above the ground with an initial speed of 20 meters/second. The height, \( h \), as a function of the horizontal distance traveled, \( d \), is given by:

\[ h = 0.75d - 0.0192d^2 \]

(a) Graph the path the ball follows.

(b) When the ball hits the ground, how far is it from the spot where the football player kicked it?

(c) What is the maximum height the ball reaches during its flight?

(d) What is the horizontal distance the ball has traveled when it reaches its maximum height?

35. A ballet dancer jumps in the air. The height, \( h(t) \), in feet, of the dancer at time \( t \), in seconds since the start of the jump, is given by:

\[ h(t) = -16t^2 + 16t \]

where \( T \) is the total time in seconds that the ballet dancer is in the air.

(a) Why does this model apply only for \( 0 \leq t \leq T \)?

(b) When, in terms of \( T \), does the maximum height of the jump occur?

(c) Show that the time, \( T \), that the dancer is in the air is related to \( H \), the maximum height of the jump, by the equation

\[ H = 4T^2 \]